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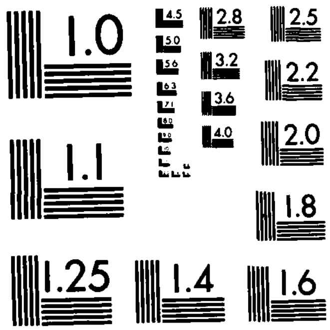

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THE APPLICATION OF THE FINITE ELEMENT TECHNIQUE TO POTENTIAL FLOW PROBLEMS: PART III

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NOMENCLATURE

A_1, A_2	objects under consideration
B_{in}, B_{out}	points interior and exterior, respectively, to the region
D	solution domain
$f(t)$	space constant
$\bar{i}, \bar{j}, \bar{k}$	unit (Cartesian) base vectors in x,y,z directions, respectively
$\bar{n}$	unit outward normal
$\bar{n}_1, \bar{n}_2, \bar{n}_0$	unit outward normal to objects A_1, A_2, and to S_0, respectively
n_x, n_y	x,y components of $\bar{n}$
$(n_x)_1, (n_y)_1$	x,y components of $\bar{n}_1$
$(n_x)_2, (n_y)_2$	x,y components of $\bar{n}_2$
$(n_x)_0, (n_y)_0$	x,y components of $\bar{n}_0$
Oxy	fixed Cartesian frame of reference
p	pressure at any point in the fluid
p_a	pressure related to acceleration
P_o	constant (pressure)
p_v	pressure related to velocity
$\bar{q}$	velocity vector
$\bar{q}_{S_1}^{n}, \bar{q}_{S_2}^{n}$	velocity of fluid adjacent to S_1, S_2 in the directions $\bar{n}_1, \bar{n}_2$, respectively
$\bar{q}_{uniform}$	velocity vector of the uniform flow exterior to S_0
$\bar{s}$	unit tangential vector
$\bar{s}_1, \bar{s}_2$	unit vectors tangential to S_1, S_2
S, S_0	curves enclosing the domain D
S_1, S_2	boundaries of the objects A_1, A_2, respectively

$t, \Delta t$	time, increment in the time t, respectively
u, v	x,y components of $\bar{q}$
$\bar{u}_1, \bar{u}_2$	velocity vectors of objects A_1, A_2
U, V	x,y components of $\bar{q}_{uniform}$
$\bar{\nabla}$	Laplace operator
$\bar{\zeta}$	vorticity
ρ	density of the fluid
ϕ	velocity potential
$\chi(v)$	functional, in terms of the function v
ψ	stream function
ψ^*, ψ_1, ψ_2	harmonic functions
ψ_P	ψ evaluated at the point P

ABSTRACT

For unsteady, irrotational flow governed by Laplace's equation, velocity potential and stream function solutions are presented with particular consideration being given to the boundary conditions. The analyses are applicable to multiple bodies each moving in an arbitrary direction with varying velocity. In the final section of the report, the problem of calculating the entrained mass for a body of arbitrary shape is considered.

1. INTRODUCTION

In previous reports [1,2] the application of the finite element technique to steady, irrotational, incompressible flow fields was considered. The special boundary conditions encountered in potential flow problems, such as the Kutta condition on an aerofoil were dealt with in some detail.

This present report extends this consideration to unsteady flow fields, concentrating particularly on potential flow problems. The scope is restricted to two dimensions. The extension to three dimensions is under investigation.

2. UNSTEADY FLOW ANALYSIS

In an unsteady field, the boundary conditions vary with time. Since potential fields are uniquely specified by the boundary conditions, there is a corresponding variation with time in the field. The potential at a point in the domain changes as signals reach that position, from the boundaries. The velocity of these signals is the acoustic velocity, which in an ideal fluid, is infinite in magnitude. Thus for potential flow, changes in the boundary conditions affect immediately all points in the field. The stream function and velocity potential distributions are thus always in accord with the boundary conditions at any given instant. This does not mean, however, that steady state conditions exist at every instant of time. The ideal fluid has inertia, and there is thus a corresponding acceleration term which must be included in the equations of motion. The solution of these equations will thus be different from that obtained using the steady-state equations, even for

identical boundary conditions.

As in steady flow problems, it is desirable to formulate the field equation in terms of a single scalar variable. This is simply done by means of the velocity potential or the stream function. There are some difficulties in the boundary conditions that arise if the stream function is used, and these are discussed in Section 3.2. For constant-density flows with vorticity, an *acceleration potential* can be derived which satisfies the Poisson field equation. This approach is particularly valuable since it can be applied to rotational flows. This report, however, deals only with the case of the irrotational field.

2.1 The Velocity Potential

For unsteady, irrotational flow, a velocity potential ϕ can be defined from the consideration of zero vorticity as follows.

The vorticity $\bar{\zeta}$ is defined by

$$\bar{\zeta} = \bar{\nabla} \times \bar{q} \quad , \tag{2.1}$$

where $\bar{q}$ is the velocity vector, which in terms of its x,y, components, u,v, respectively, is given by

$$\bar{q} = u\bar{i} + v\bar{j} \quad , \tag{2.2}$$

and where $\bar{\nabla}$ is the Laplace operator. In Eq. (2.2), $\bar{i}$ and $\bar{j}$ are the unit base vectors in the x and y direction, respectively.

The condition of zero vorticity, i.e.,

$$\bar{\zeta} = \bar{0} \quad , \tag{2.3}$$

becomes from Eq. (2.1)

$$\bar{\nabla} \times \bar{q} = \bar{0} \quad . \tag{2.4}$$

Equation (2.4) may be satisfied by defining the velocity vector, $\bar{q}$, in terms of a scalar ϕ as*

$$\bar{q} = - \bar{\nabla}\phi \quad . \tag{2.5}$$

The scalar ϕ in Eq. (2.5) is known as the velocity potential. From Eq. (2.5), the velocity components u,v, respectively, become

$$u = - \frac{\partial \phi}{\partial x} \quad , \tag{2.6a}$$

$$v = - \frac{\partial \phi}{\partial y} \quad . \tag{2.6b}$$

If, in addition, the fluid is assumed to be incompressible, then it can be shown [3,4] that

$$\bar{\nabla} \cdot \bar{q} = 0 \quad . \tag{2.7}$$

Substituting Eq. (2.5) into Eq. (2.7) results in

$$- \bar{\nabla} . (\bar{\nabla}\phi) = 0 \quad , \tag{2.8a}$$

which further reduces to

$$\nabla^2 \phi = 0 \quad . \tag{2.8b}$$

The combination of the irrotationality and continuity conditions therefore yields Laplace's equation in ϕ. Although the

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* The convention that $\bar{q} = - \bar{\nabla}\phi$ is adopted in this report. However, some authors prefer the convention $\bar{q} = + \bar{\nabla}\phi$.

independent variable t may appear in the expressions for the velocity vector $\bar{q}$, and consequently in the velocity potential ϕ, the above analysis shows that the field equation for the unsteady, incompressible, irrotational flow is the same as that for the steady case. The boundary conditions are, however, the appropriate *unsteady* ones, and are dealt with in Section 3.1.

The pressure at any point in the fluid, p, may be obtained from the generalized Bernouilli's equation [3,4]

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}(\bar{\nabla}\phi)^2 + \frac{p}{\rho} = f(t) \quad , \tag{2.9}$$

where ρ is the density of the fluid, and f(t) is a space constant. It is noted that for the unsteady case both space and time derivatives of the velocity potential ϕ are required. The space constant f(t) in Eq. (2.9) allows for a change in the absolute pressure by an arbitrary variation of the external pressure on the system. If the external pressure is considered fixed, then f(t) reduces to a simple constant, both in space and time.

Although the velocity potential ϕ is of importance in the analysis of both steady and unsteady flows, it is rather restrictive in that it can only be used if the flow is irrotational.

2.2 The Stream Function

Laplace's equation similarly applies to the stream function ψ in unsteady, irrotational flow, if the fluid is incompressible. This may be shown as follows.

The condition of incompressibility, i.e. Eq. (2.7),

$$\bar{\nabla}\cdot\bar{q} = 0 \quad , \tag{2.7}$$

may be satisfied by choosing the velocity vector $\bar{q}$ as

$$\bar{q} = -\frac{\partial\psi}{\partial y}\bar{i} + \frac{\partial\psi}{\partial x}\bar{j} \quad . \tag{2.10}$$

From Eq. (2.10), the velocity components u,v, respectively, become

$$u = -\frac{\partial\psi}{\partial y} \quad , \qquad (2$$

$$v = \frac{\partial\psi}{\partial x} \quad . \tag{2.11b}$$

Using the condition of irrotationality, Eq. (2.4), there is obtained

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \quad , \tag{2.12a}$$

which is Laplace's equation in ψ, i.e.,

$$\nabla^2\psi = 0 \quad . \tag{2.12b}$$

The statement made regarding the field equation for the velocity potential ϕ in Section 2.1 similarly applies to the stream function ψ, irregardless of whether the independent variable t appears explicitly or not in the velocity vector $\bar{q}$ and (consequently) in the stream function ψ.

The finite element technique may be applied to solve for ψ in the manner described in Section 4.2. The boundary conditions will generally be more complex than those for the velocity potential. These

conditions are considered in detail in Section 3.2.

2.3 The Cauchy-Riemann Conditions

It was shown in Sections 2.1 and 2.2 that for unsteady, incompressible, irrotational flow, both a velocity potential ϕ, and a stream function ψ exist, and furthermore that both these scalar fields ϕ and ψ satisfy Laplace's equation.

It is instructive to note that although the unsteady flow is being considered, i.e.,

$$\phi = \phi(x,y,t) \quad , \tag{2.13a}$$

$$\psi = \psi(x,y,t) \quad , \tag{2.13b}$$

the Cauchy-Riemann conditions still apply. This may be shown as follows, where throughout, the explicit dependence of the dependent variables $\bar{q}$, u, v, ϕ, and ψ on the independent variables x, y, and t, is included. From Eq. (2.2) there is obtained

$$\bar{q}(x,y,t) = u(x,y,t)\bar{i} + v(x,y,t)\bar{j} \quad . \tag{2.14}$$

Equations (2.6a) and (2.6b) similarly reduce to

$$u(x,y,t) = -\frac{\partial\phi(x,y,t)}{\partial x} \quad , \tag{2.15a}$$

and

$$v(x,y,t) = -\frac{\partial\phi(x,y,t)}{\partial y} \quad . \tag{2.15b}$$

Furthermore, from Eqs. (2.11a) and (2.11b), it follows that

$$u(x,y,t) = -\frac{\partial\psi(x,y,t)}{\partial y} , \tag{2.16a}$$

$$v(x,y,t) = \frac{\partial\psi(x,y,t)}{\partial x} . \tag{2.16b}$$

Comparing the set of equations (2.15) with the set (2.16) results in

$$\frac{\partial\phi(x,y,t)}{\partial x} = \frac{\partial\psi(x,y,t)}{\partial y} , \tag{2.17a}$$

$$\frac{\partial\phi(x,y,t)}{\partial y} = -\frac{\partial\psi(x,y,t)}{\partial y} , \tag{2.17b}$$

which shows that although ϕ and ψ are dependent on the variable t, they still satisfy the Cauchy-Riemann conditions. The above analysis can be extended to show that, in addition to the set of Eqs. (2.17), the following relations hold,

$$\frac{\partial\phi(x,y,t)}{\partial n} = \frac{\partial\psi(x,y,t)}{\partial s} , \tag{2.18a}$$

$$\frac{\partial\phi(x,y,t)}{\partial s} = -\frac{\partial\psi(x,y,t)}{\partial n} , \tag{2.18b}$$

where $-\frac{\partial\phi}{\partial n}$ and $-\frac{\partial\phi}{\partial s}$ are the velocities in the directions of increasing $\bar{n}$ and $\bar{s}$, respectively. The vectors $\bar{n}$ and $\bar{s}$ are mutually orthogonal and may, for example, be chosen as the unit outward normal to the curve S, and as the unit vector tangential to S, respectively, see Figure 2.1.

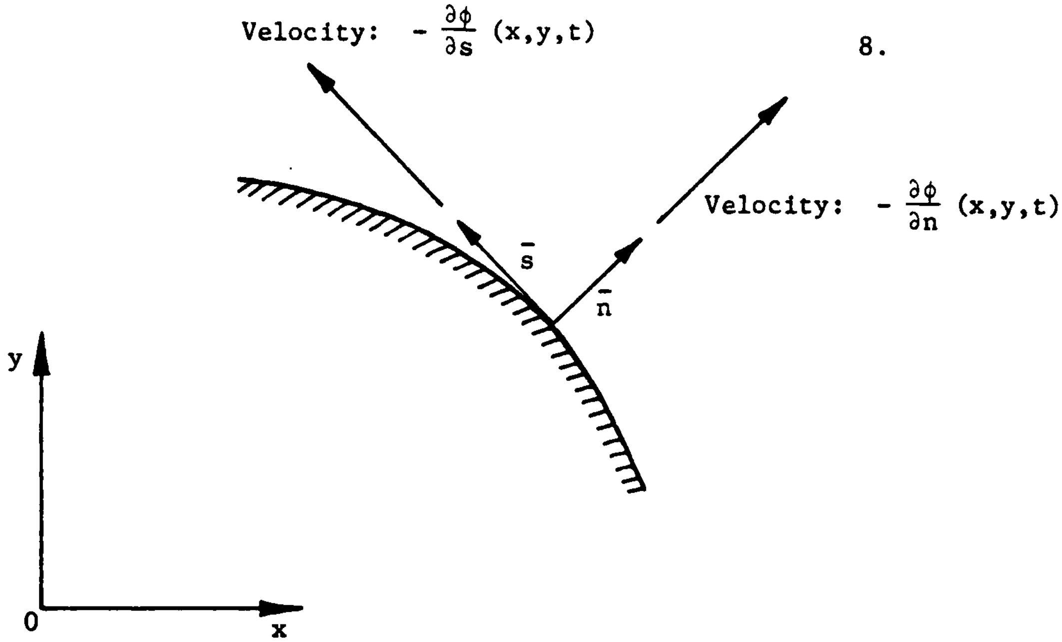

Fig. 2.1 Velocity components in the directions $\bar{n}$ and $\bar{s}$.

2.4 The Acceleration Potential

A potential function P can be defined for acceleration, where the flow is such that the vorticity is preserved [3]. Its special importance is that it may be defined for *both* rotational and irrotational flows, whereas the velocity potential can be defined for the latter case only, as had already been mentioned in the preceding Section 2.1. The relevant field equation in terms of P will not be derived here, but it can be shown to satisfy Poisson's equation. The analysis in terms of the acceleration potential P is especially useful in the analysis of discontinuous flows since, although the velocity is discontinuous across a vortex sheet, and the velocity potential ϕ may *also* be discontinuous there, the pressure and hence the acceleration potential P are continuous across such a sheet.

The authors are presently investigating the application of the finite element technique to such discontinuous flows.

3. BOUNDARY CONDITIONS

For a stationary impervious boundary, the boundary condition is of the homogeneous Neumann type [5, 6], but for a moving boundary (whether that of a rigid surface, a deforming surface, or a two-phase surface) the kinematic boundary condition will be of the nonhomogeneous Neumann type as is shown in the subsequent sections. In the following, only irrotational flow is considered.

3.1 The Velocity Potential

In considering the boundary conditions for the velocity potential, it is best to consider a particular example. The same example will be used again when dealing with the boundary conditions for the stream function ψ in Section 3.2.

The example of interest is that as shown in Figure 3.1, namely two rigid objects in motion† in an otherwise uniform stream.

† In this and the following analysis, the velocities of the objects may depend explicitly on the time t.

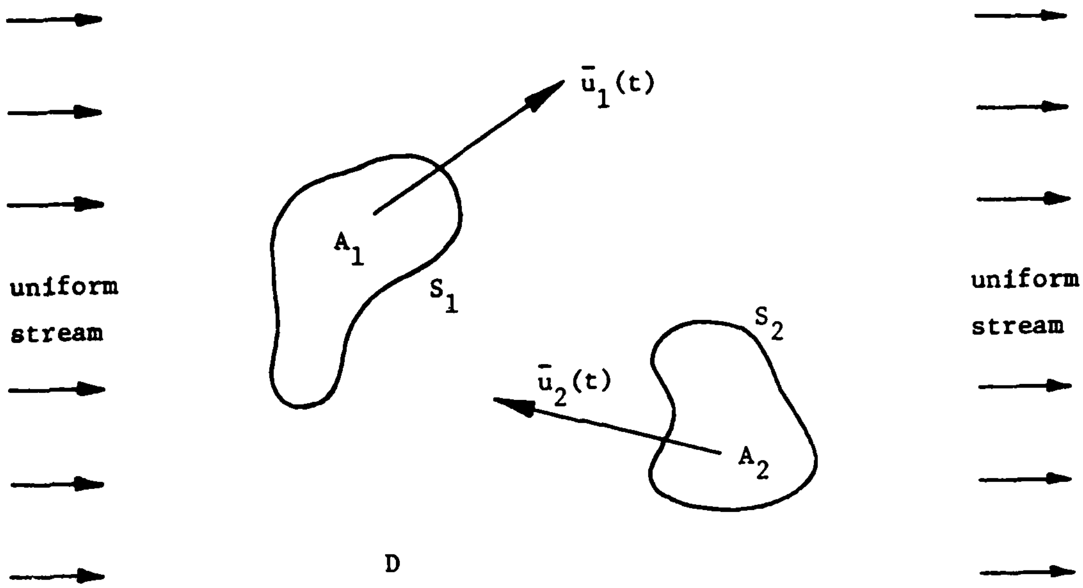

Fig. 3.1 Two moving objects in an otherwise uniform stream.

Let the two objects be denoted by A_1 and A_2 as shown. Furthermore let the boundaries of A_1, A_2 be given by the (closed) curves S_1, S_2, respectively, and let their corresponding velocities be given by $\bar{u}_1$ and $\bar{u}_2$. It is assumed in the analysis presented throughout this report that the objects are impervious, consequently there can be no flow across the boundaries S_1 and S_2.

For purposes of illustration, the following analysis only considers one object, which can then be generalized to two or more objects moving in the same flow field. Consider therefore the case illustrated in Figure 3.2, where the object, denoted by A_1 and bounded by S_1, is shown at its position at time t, i.e., with respect to the

fixed coordinate frame Oxy.

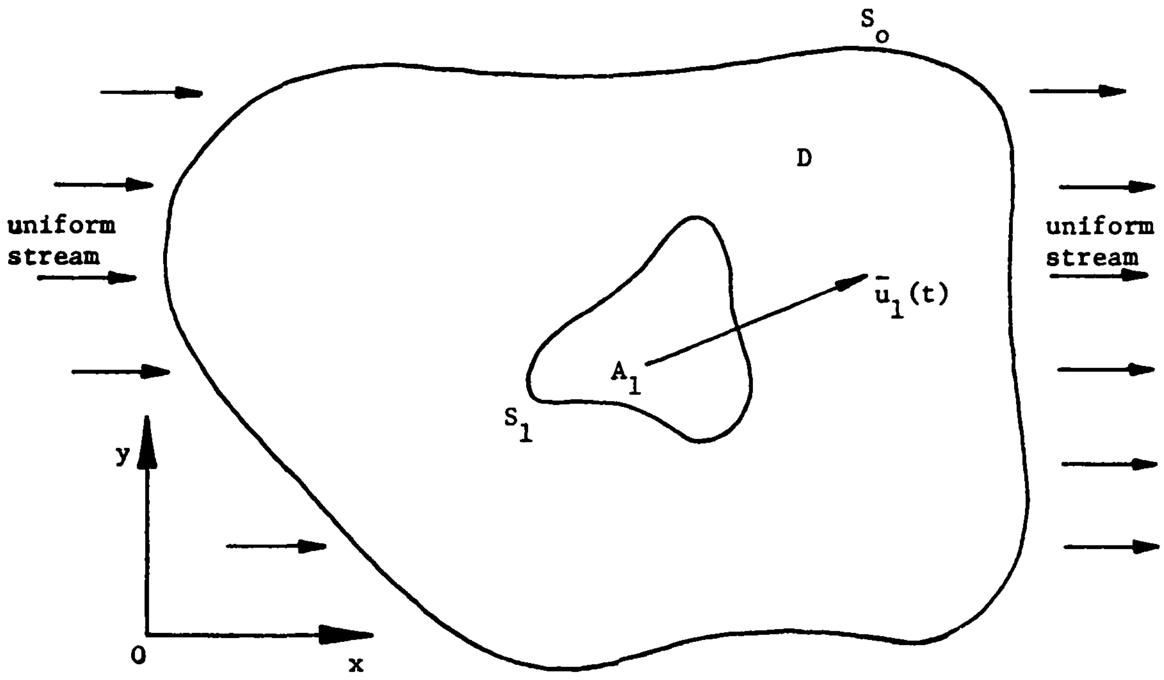

Fig. 3.2 Object A_1 moving in an otherwise uniform stream.

Let the outer boundary of the control volume being considered be denoted by S_0, which is assumed to be sufficiently far removed from A_1, and hence from S_1, such that any movement of the object A_1 does not affect the uniform stream exterior to S_0. Acceleration of the object A_1 is allowable in this analysis, and hence the velocity of A_1 is indicated as $\bar{u}_1(t)$, see Figure 3.2.

The finite element analysis presented in this report considers the solution at fixed instants of time. Suppose the solution is sought at time t, i.e., when the object is, with respect to the chosen axes Oxy, at its location as shown in Figure 3.2. Let the unit outward normal to S_1 be denoted by $\bar{n}_1$, then from the condition that no flow occur across this boundary (since S_1 is impervious), it is clear

that the normal velocity of the boundary of the object must be the same as the normal component of the velocity of the fluid which is adjacent to this boundary. The velocity of the *object* in the direction $\bar{n}_1$, which will be denoted by $\bar{u}_1^{\,n}$, is given by

$$\bar{u}_1^{\,n} = \bar{u}_1 \cdot \bar{n}_1 \quad . \tag{3.1}$$

If the velocity of the *fluid* in this same direction and along the same boundary S_1 is denoted by $\bar{q}_{S_1}^{\,n}$, then it follows from $\bar{q}_{S_1}^{\,n} = \bar{u}_1^{n}$ that

$$\bar{q}_{S_1}^{\,n} = \bar{u}_1 \cdot \bar{n}_1 \quad . \tag{3.2}$$

From Eqs. (2.18a) and (2.18b), it is known that $\bar{q}_{S_1}^{\,n}$ may also be written as

$$\bar{q}_{S_1}^{\,n} = -\frac{\partial \phi}{\partial n} \quad \text{on } S_1 \quad . \tag{3.3}$$

Substitution of Eq. (3.2) into Eq. (3.3) results in

$$\frac{\partial \phi}{\partial n} = -\bar{u}_1 \cdot \bar{n}_1 \quad \text{on } S_1 \quad . \tag{3.4}$$

It is important to note that the boundary condition given by Eq. (3.4) is of the nonhomogeneous Neumann type. This is in contrast to the homogeneous boundary condition $\frac{\partial \phi}{\partial n} = 0$ on S_1, which occurs when A_1 is stationary [6].

Since the flow exterior to S_0 is assumed to be uniform at all times, it follows that on S_0 the velocity potential ϕ is known, at least apart from an arbitrary constant of integration, consequently on S_0 a Dirichlet condition exists for all times, as is shown below.

Since the flow exterior to, as well as on S_0, is uniform, it follows that

$$\left.\begin{aligned} u(x,y,t) &= U \\ v(x,y,t) &= V \end{aligned}\right\} \text{ on } S_0 \quad , \tag{3.5}$$

where U, V are the magnitudes of the uniform velocity in the x,y directions, respectively. From Eqs. (2.15a) and (2.15b) it follows that

$$\left.\begin{aligned} \frac{\partial\phi}{\partial x} &= -U \\ \frac{\partial\phi}{\partial y} &= -V \end{aligned}\right\} \text{ on } S_0 \quad , \tag{3.6a}$$

and furthermore, the condition that the flow is uniform gives

$$\frac{\partial\phi}{\partial t} = 0 \quad , \tag{3.6b}$$

with solution

$$\phi = -Ux - Vy + C \quad \text{on } S_0 \quad , \tag{3.7}$$

where C is an arbitrary constant of integration which may, without loss of generality†, be chosen to be zero. Consequently the Dirichlet condition for ϕ on S_0 reduces to

$$\phi = -Ux - Vy \quad \text{on } S_0 \quad . \tag{3.8}$$

The boundary condition for ϕ on S_0 could have been derived equally as well as a nonhomogeneous Neumann boundary condition as

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† The choice C = 0 *also* corresponds to a change in position of the coordinate frame Oxy, which is arbitrary.

follows. Noting from Eq. (3.6b) that $\frac{\partial \phi}{\partial t} = 0$, it follows that

$$\frac{\partial \phi}{\partial n} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial n} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial n} \quad \text{on } S_0 \quad . \tag{3.9}$$

Since from Figure 3.3,

$$\left.\begin{aligned} (n_x)_0 &= \frac{\partial x}{\partial n} \quad &(3.10a) \\ (n_y)_0 &= \frac{\partial y}{\partial n} \quad &(3.10b) \end{aligned}\right\} \text{ on } S_0 \quad ,$$

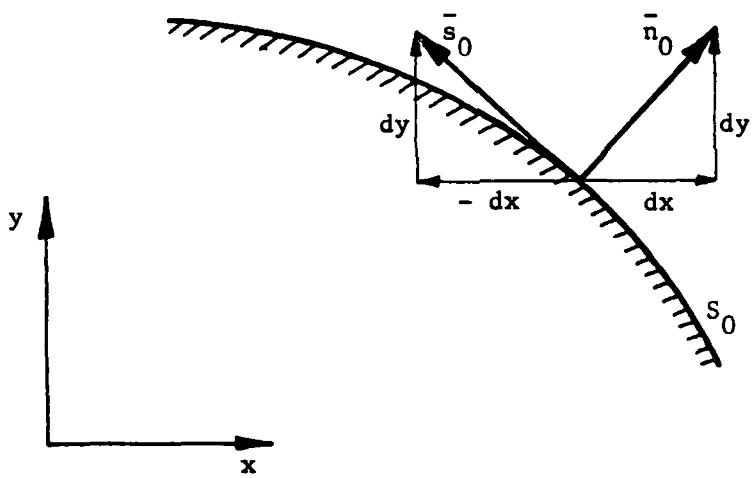

Fig. 3.3 The normal and tangential vectors $\bar{n}_0$ and $\bar{s}_0$ on the boundary S_0.

where $(n_x)_0$ and $(n_y)_0$ are the x and y components of the unit outward normal to S_0, $\bar{n}_0$, Eq. (3.9) reduces to

$$\frac{\partial \phi}{\partial n} = \frac{\partial \phi}{\partial x}(n_x)_0 + \frac{\partial \phi}{\partial y}(n_y)_0 \quad \text{on } S_0 \quad . \tag{3.11}$$

By using the set of equations (3.6a), Eq. (3.11) may be reduced further to

$$\frac{\partial\phi}{\partial n} = -U(n_x)_0 - V(n_y)_0 \quad \text{on } S_0 \quad . \tag{3.12}$$

It is *always* possible to choose the coordinate frame Oxy in such a way that either U or V is zero, and furthermore the boundary S_0 may be chosen as a rectangle so that in addition to U or V being zero, the components $(n_x)_0$ and $(n_y)_0$ become simply either +1 or -1, which simplifies the calculations. In passing it is noted that for an arbitrary shaped curve S_0, the corresponding boundary condition to Eq. (3.12) can be written as

$$\frac{\partial\phi}{\partial n} = -\bar{q}_{uniform} \cdot \bar{n}_0 \quad \text{on } S_0 \quad , \tag{3.13}$$

where $\bar{q}_{uniform}$ is the velocity vector of the uniform flow exterior to S_0, i.e.,

$$\bar{q}_{uniform} = U\bar{i} + V\bar{j} \quad . \tag{3.14}$$

It was shown previously that on S_1 the non-homogeneous Neumann condition, Eq. (3.4), must be satisfied. If the velocity of the object A_1 is prescribed, then its location, as well as its unit outward normal $\bar{n}_1$, is prescribed at all times with respect to Oxy. Consequently at every instant of time Eq. (3.4) is determined, and hence also the boundary condition on S_1.

In summary then, the flow field, in terms of the velocity potential ϕ, may be obtained for any time t as the solution (see

Section 2), to

$$\nabla^2\phi = 0 \quad \text{in } D \quad , \tag{2.8b}$$

subject to the Dirichlet condition

$$\phi = -Ux - Vy \quad \text{on } S_0 \quad , \tag{3.8}$$

or the nonhomogeneous Neumann condition

$$\frac{\partial\phi}{\partial n} = -\bar{q}_{uniform} \cdot \bar{n}_0 \quad \text{on } S_0 \quad , \tag{3.13}$$

and the nonhomogeneous Neumann condition

$$\frac{\partial\phi}{\partial n} = -\bar{u}_1 \cdot \bar{n}_1 \quad \text{on } S_1 \quad . \tag{3.4}$$

It is noted that at any given time $\bar{u}_1$ is prescribed and the location and orientation of A_1 is fixed and hence $\bar{n}_1$ is prescribed, consequently the boundary condition on S_1, see Eq. (3.4), is completely prescribed. For additional bodies, Eq. (3.4) is applicable also.

3.2 <u>The Stream Function</u>

Consider again the example used in Section 3.1, namely the unsteady, irrotational flow field due to a uniform stream of an incompressible fluid obstructed by moving objects, as shown in Figure 3.2. Arguments similar to those used in that section indicate that ψ is completely prescribed on S_0 at least up to an arbitrary constant of integration, which, as was indicated in Section 3.1, may be chosen to be zero. This procedure is illustrated as follows. The boundary condition for ψ on S_0 is obtained from the conditions, see Eq. (3.5),

$$\left.\begin{aligned} u(x,y,t) &= U \\ v(x,y,t) &= V \end{aligned}\right\} \text{ on } S_0 \quad , \tag{3.5}$$

which in view of Eqs. (2.16a) and (2.16b) result in

$$\psi = Vx - Uy + C \quad \text{on } S_0 \quad . \tag{3.15}$$

In Eq. (3.15) the arbitrary constant of integration is denoted by C, which may be chosen to be equal to zero, thus reducing Eq. (3.15) to

$$\psi = Vx - Uy \quad \text{on } S_0 \quad . \tag{3.16}$$

In a similar fashion to that discussed for the velocity potential ϕ, the boundary condition for ψ on S_0 may be written as a nonhomogeneous Neumann condition. This may be shown to be

$$\frac{\partial\psi}{\partial n} = V(n_x)_0 - U(n_y)_0 \quad \text{on } S_0 \quad , \tag{3.17a}$$

or

$$\frac{\partial\psi}{\partial n} = -(\bar{q}_{uniform} \times \bar{n}_0) \cdot \bar{k} \quad \text{on } S_0 \quad , \tag{3.17b}$$

where $\bar{k}$ is the unit base vector in the z direction, perpendicular to both $\bar{i}$ and $\bar{j}$, see Figure 3.4.

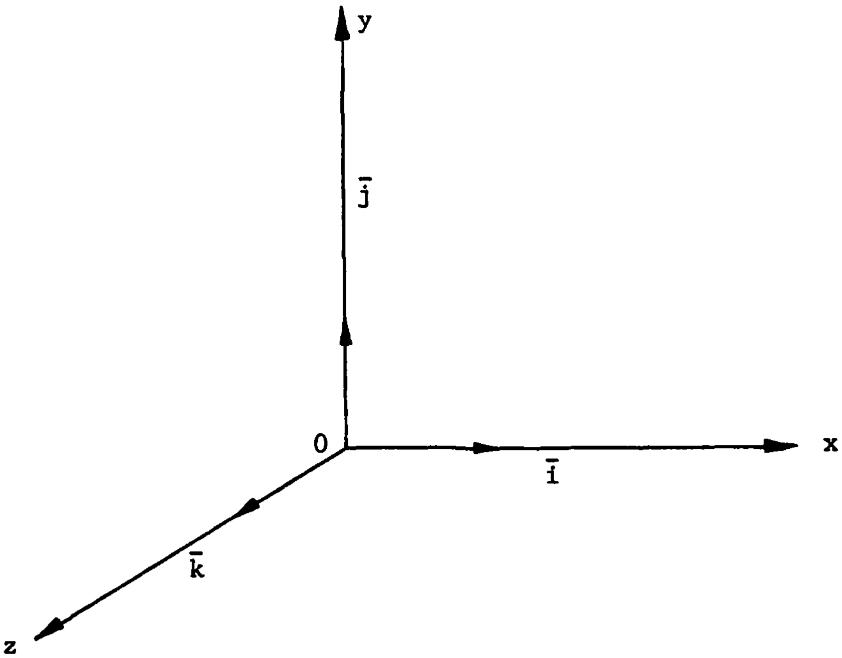

Fig. 3.4 The base vectors $\bar{i}$, $\bar{j}$, and $\bar{k}$.

The other boundary condition to be investigated is that on the boundary S_1 of the object A_1. It is worth reiterating at this point what happens on this boundary in the steady* case. In the steady case, the stream function $\psi = \psi(x,y)$ may simply be written as

$$\psi = \psi(s) \quad \text{on } S_1 \quad , \tag{3.18}$$

where s is measured along S_1 from some arbitrary datum, i.e., from some arbitrarily chosen point P, where s = 0. The condition that no flow occur across the impervious boundary S_1 dictates that the velocity normal to S_1 must be zero. Consequently, from Eq. (2.18a) there obtains

$$\frac{\partial\psi(s)}{\partial s} = 0 \quad \text{on } S_1 \quad , \tag{3.19}$$

and hence

* i.e., when the object A_1 is stationary.

$$\psi = \text{constant} \quad \text{on } S_1 \quad . \tag{3.20}$$

It needs pointing out, that although ψ is constant on S_1 in the steady case, this constant is not known in value. However, it may be obtained by a further consideration of the boundary conditions on S_0 as is outlined in the following sections, as well as in the literature [2,7].

Extending the above argument to unsteady flow, it is clear that at any *given instant* of time t the stream function ψ must still be a function of s, i.e.,

$$\psi = \psi(s) \quad \text{on } S_1 \quad , \tag{3.21}$$

at that particular instant of time. In general, it would be more appropriate to make the statement that

$$\psi = \psi(s,t) \quad \text{on } S_1 \quad , \tag{3.22}$$

since as time varies, the location of S_1 is changing accordingly. It was pointed out in Section 3.1, however, that in the present approach the finite element solution is to be obtained at a fixed instant of time, hence the formulation (3.21) will suffice.

Since the object A_1 moves with time, the fluid adjacent to S_1 must have a velocity associated with it and consequently the expression (3.19) no longer holds. From Eq. (3.21), there obtains

$$d\psi(s) = \frac{d\psi}{ds}\, ds \quad \text{on } S_1 \quad . \tag{3.23}$$

Integrating Eq. (3.23) yields

$$\psi(s) = \psi(0) + \int_0^s \frac{d\psi}{ds}\, ds \quad \text{on } S_1 \quad . \tag{3.24}$$

Using the relation (2.18a), i.e.,

$$\frac{\partial\phi}{\partial n} = \frac{\partial\psi}{\partial s} \quad , \tag{2.18a}$$

and noting that $\frac{d\psi}{ds} = \frac{\partial\psi}{\partial s}$ for the case being considered, Eq. (3.24) may be written as

$$\psi(s) = \psi(0) + \int_0^s \frac{\partial\phi}{\partial n}\, ds \quad \text{on } S_1 \quad . \tag{3.25}$$

Substituting Eq. (3.4) into Eq. (3.25) results in

$$\psi(s) = \psi(0) - \int_0^s \bar{u}_1 \cdot \bar{n}_1\, ds \quad \text{on } S_1 \quad . \tag{3.26}$$

Equation (3.26) therefore corresponds to a Dirichlet condition for ψ on S_1. The only quantity on the right-hand side of Eq. (3.26) which is still unknown at this time is the constant value $\psi(0)$, whose determination is discussed below.

3.2.1 Determination of the Arbitrary Constant

To determine the constant $\psi(0)$, consider Figure 3.5.

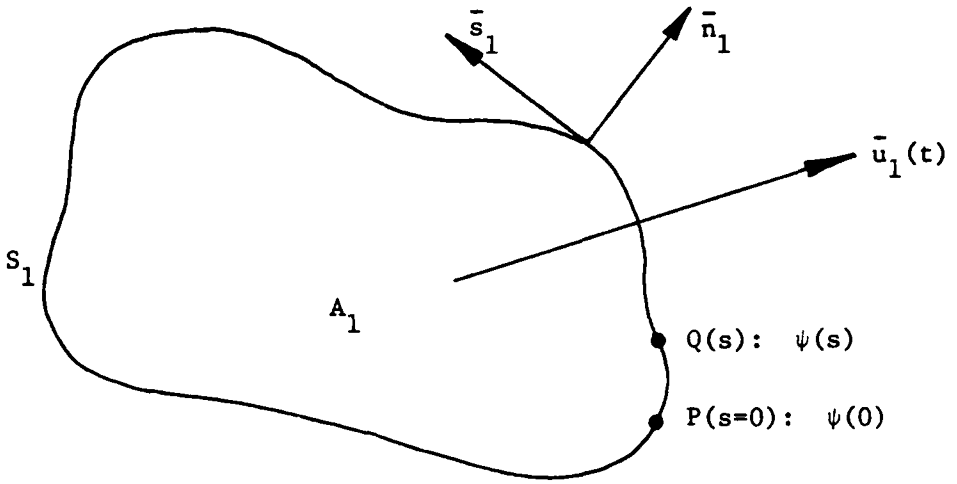

Fig. 3.5 Object A_1 bounded by the curve S_1.

Let s = 0 at an arbitrarily chosen point P on S_1, and let Q be another point on S_1, a distance s away from P and which is measured along S_1 in a counter-clockwise direction. Furthermore let the stream function be denoted by $\psi(0)$ and $\psi(s)$ at P and Q, respectively.

In considering the Dirichlet condition (3.26), let it be assumed that $\psi(0)$ at P is given by the constant a_1. It is clear that, through Eq. (3.26), $\psi(s)$ is completely described on S_1 provided the constant a_1 is known. This constant may be evaluated as follows. Consider Figure 3.6.

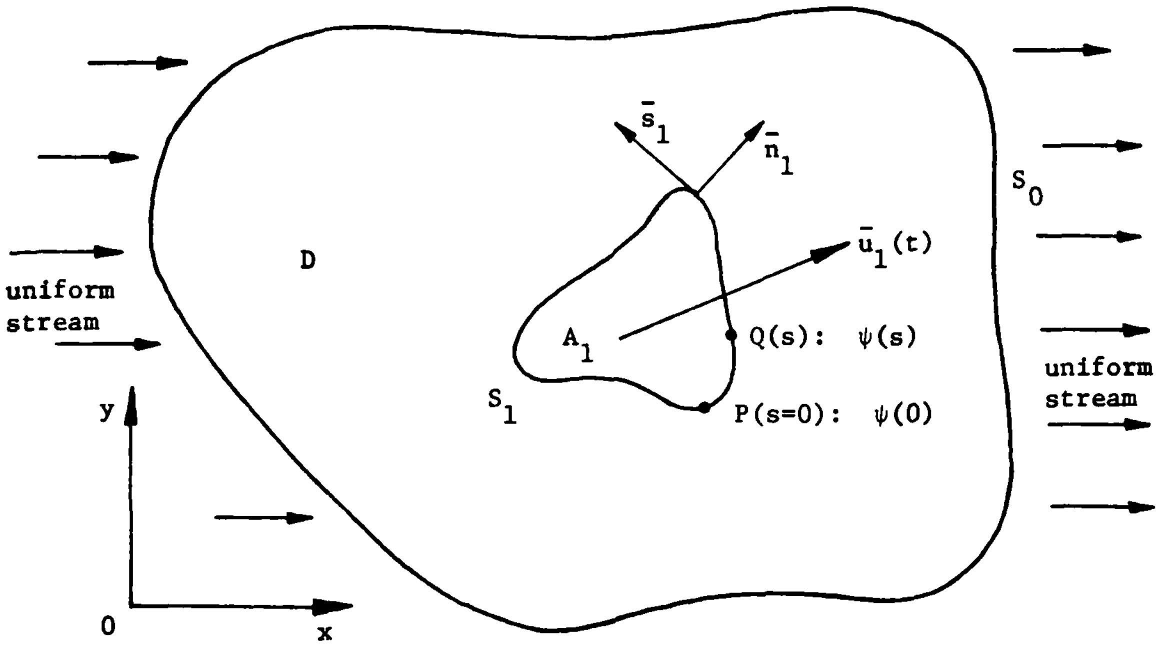

Fig. 3.6 Object A_1 moving in an otherwise uniform stream.

Let a new stream function ψ^* be defined by

$$\psi^* = \psi_1 + a_1\psi_2 \quad , \tag{3.27}$$

where ψ_1 and ψ_2 are defined by the following subproblems.

1. Find ψ_1 such that

$$\nabla^2\psi_1 = 0 \quad \text{in } D \quad , \tag{3.28}$$

subject to the Dirichlet condition

$$\psi_1 = Vx - Uy \quad \text{on } S_0 \quad , \tag{3.29a}$$

or the nonhomogeneous Neumann condition

$$\frac{\partial \psi_1}{\partial n} = -(\bar{q}_{uniform} \times \bar{n}_0) \cdot \bar{k} \quad \text{on } S_0 \quad , \tag{3.29b}$$

and the Dirichlet condition

$$\psi_1 = -\int_0^s \bar{u}_1 \cdot \bar{n}_1 \, ds \quad \text{on } S_1 \quad . \tag{3.30}$$

2. Find ψ_2 such that

$$\nabla^2 \psi_2 = 0 \quad \text{in D} \quad , \tag{3.31}$$

subject to the Dirichlet conditions

$$\psi_2 = 0 \quad \text{on } S_0 \quad , \tag{3.32}$$

$$\psi_2 = 1 \quad \text{on } S_1 \quad . \tag{3.33}$$

From Eq. (3.27), and the subproblems 1. and 2. as defined above, this new stream function ψ^* is seen to be the solution to

$$\nabla^2 \psi^* = 0 \quad \text{in D} \quad , \tag{3.34}$$

subject to the Dirichlet condition

$$\psi^* = Vx - Uy \quad \text{on } S_0 \quad , \tag{3.35a}$$

or the nonhomogeneous Neumann condition

$$\frac{\partial \psi^*}{\partial n} = -(\bar{q}_{uniform} \times \bar{n}_0) \cdot \bar{k} \quad \text{on } S_0 \quad , \tag{3.35b}$$

and the Dirichlet condition

$$\psi^* = a_1 - \int_0^s \bar{u}_1 \cdot \bar{n}_1 \, dS \quad \text{on } S_1 \quad . \tag{3.36}$$

Comparing Eqs. (3.34), (3.35a), (3.35b), and (3.36) with Eqs. (2.12b), (3.16), (3.17b), and (3.26), and noting that $\psi(0) = a_1$, it is clear that ψ^* as defined by Eq. (3.27),

$$\psi^* = \psi_1 + a_1\psi_2 \quad , \tag{3.27}$$

is the solution ψ to the problem at hand.

Subproblems 1. and 2. are completely defined and hence can be solved using the finite element method. The constant a_1 in Eq. (3.27) is determined as follows. Choose an arbitrary point in D, however close to S_0, say B_{in}. Then, since the flow is uniform at S_0, it is expected that the flow is still uniform at this newly chosen point B_{in}, *provided* it is close to S_0. Consequently the value for the stream function at B_{in} must be the same as that for the point B_{out} in the outer field near S_0, provided that B_{in} and B_{out} lie on the same streamline, see Figure 3.7.

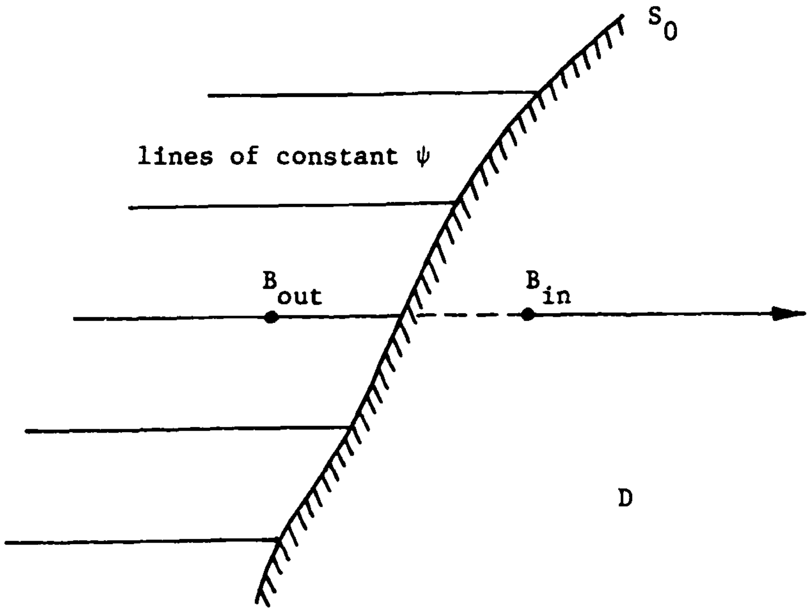

Fig. 3.7 Streamlines crossing the boundary S_0.

From Eq. (3.27), and dropping the superscript *,

$$\psi = \psi_1 + a_1\psi_2 \quad , \tag{3.37}$$

and applying this equation to the point B_{in}, there obtains

$$\psi(B_{in}) = \psi_1(B_{in}) + a_1\psi_2(B_{in}) \quad . \tag{3.38}$$

Substituting the finite element solutions for ψ_1 and ψ_2 at the point B_{in} into Eq. (3.38), and using the known value of $\psi(B_{out})$ in

$$\psi(B_{in}) = \psi(B_{out}) \quad , \tag{3.39}$$

yields after rearrangement of Eq. (3.38)

$$a_1 = \frac{\psi(B_{out}) - \psi_1(B_{in})}{\psi_2(B_{in})} \quad . \tag{3.40}$$

Equation (3.40) yields a_1 provided that $\psi_2(B_{in}) \neq 0$. It is noted from Subproblem 2. that ψ_2 is different from zero in D, and hence $\psi_2(B_{in}) \neq 0$ for B_{in} not on S_0.

In passing, it is pointed out that in the above analysis only one constant may be chosen arbitrarily, i.e., in Eq. (3.16) C was chosen to be zero. Consequently the need for the evaluation of the constant a_1 is apparent. However, if the nonhomogeneous Neumann boundary condition (3.17b) on S_0 were chosen rather than the Dirichlet condition (3.16), then there would be no need to perform the above analysis to obtain a_1. However, if more than one object is considered the choice of Eq. (3.17) rather than (3.16) does not remove this necessity to determine another constant, since an additional constant defined by $b_1 = \psi(0)$ on S_2 must be evaluated in the manner outlined above.

It is interesting to consider how the value of $a_1 = \psi(0)$ at the fixed boundary point P changes from its value at time t to its value at time $t + \Delta t$. Since for this fixed point P on the boundary S_1 of the object A_1, s was chosen to be zero, it retains this value ($s = 0$) with changes of time.

In general, from

$$\psi = \psi(s,t) \quad , \tag{3.22}$$

the differential of $\psi(s,t)$ is given by

$$d\psi(s,t) = \frac{\partial\psi}{\partial s}\,ds + \frac{\partial\psi}{\partial t}\,dt \quad . \tag{3.41}$$

At the point P, since s = 0 for all times, it follows that

$$d\psi_P(t) = d\psi(0,t) = \frac{\partial\psi(0,t)}{\partial t}\,dt \quad ,$$

and consequently

$$\psi_P(t+\Delta t) = \psi(0,t+\Delta t) = \psi(0,t) + \int_t^{t+\Delta t} \frac{\partial\psi(0,t)}{\partial t}\,dt \quad . \tag{3.42}$$

Equation (3.42) can be used if the term $\frac{\partial\psi(0,t)}{\partial t}$ is known. Alternatively, $\psi_P(t+\Delta t)$ can be determined by repeating the previous analysis, as follows. After the increment of time Δt, the object A_1 occupies its new location, which is completely determined, presuming $\bar{u}_1(t)$ is known. For this new position and this new time $t + \Delta t$, the earlier analysis can be repeated with a new constant a_1 being determined as before. This procedure can then be repeated for subsequent time intervals.

4. THE FINITE ELEMENT SOLUTION FOR UNSTEADY FLOW

As indicated in the previous sections, both a velocity potential ϕ and a stream function ψ exist for unsteady, irrotational flow, which satisfy Laplace's equation if the fluid is incompressible.

The finite element method can thus be applied in the usual manner [2], except that the boundary conditions will be of the Dirichlet as well as the nonhomogeneous Neumann type when the flow field involves moving boundaries, see Section 3. This means that the mathematical

statement of the problem for either the velocity potential ϕ or the stream function ψ at any given instant of time reduces to the following (see Figure 4.1).

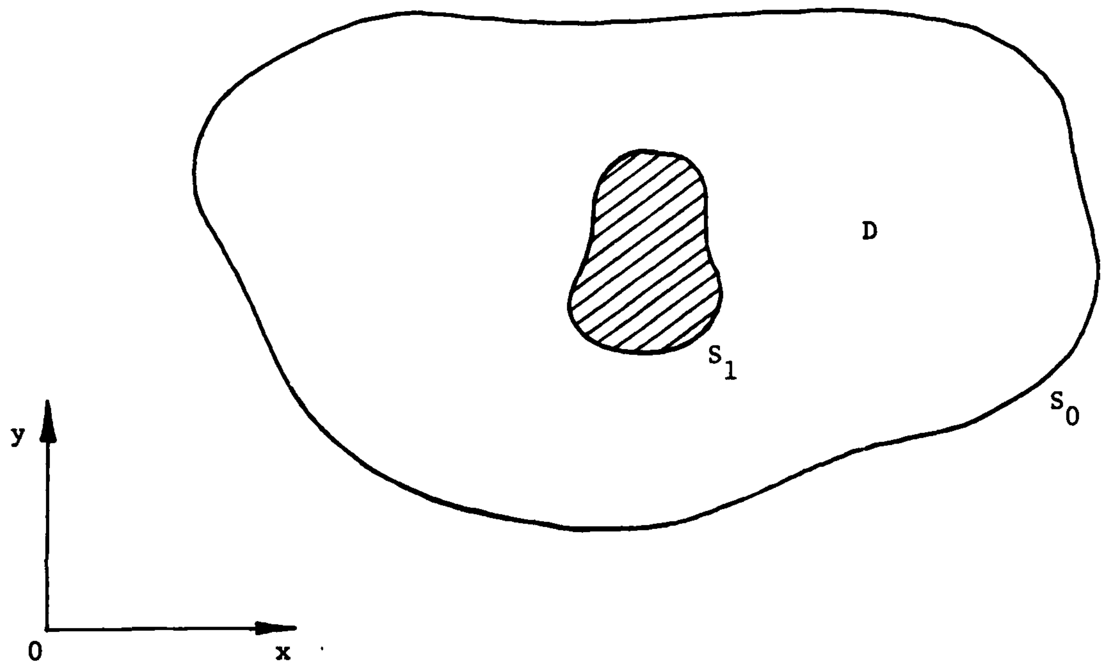

Fig. 4.1 Domain D enclosed by the surfaces S_0 and S_1.

Obtain that function $u(x,y,t_0)$, for t_0 arbitrary, which satisfies Laplace's equation in the domain D, .e.,

$$\nabla^2 u(x,y,t_0) = 0 \quad \text{in D} \quad , \tag{4.1}$$

subject to the following boundary conditions:

1. $$u(x,y,t_0) = g(x,y,t_0) \quad \text{on } S_0 \quad , \tag{4.2a}$$

or

$$\frac{\partial u(x,y,t_0)}{\partial n} = h(x,y,t_0) \quad \text{on } S_0 \quad . \tag{4.2b}$$

2. $$\frac{\partial u(x,y,t_0)}{\partial n} = z(x,y,t_0) \quad \text{on } S_1 \quad , \tag{4.3}$$

where $g(x,y,t_0)$ and $h(x,y,t_0)$ are prescribed functions of position (and time†) along the fixed (i.e., with respect to the fixed reference frame Oxy) boundary S_0, $z(x,y,t_0)$ is a prescribed function of position and time t_0 along S_1. Although the boundary S_1 moves with time, its location may be considered fixed and completely prescribed with respect to the fixed reference frame Oxy at any instant of time and hence also at t_0.

The problem stated above may be reformulated in terms of the calculus of variations as follows. Obtain that function $v(x,y,t_0)$, which minimizes the functional

$$X(v) = \int_D \tfrac{1}{2}\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]dD - A\int_{S_0} h(x,y,t_0)v\,dS_0 - \int_{S_1} z(x,y,t_0)v\,dS_1 \ , \tag{4.4}$$

where A is a constant, either 0 or 1 in value, depending on which formulation is chosen, i.e., A = 0 for the problem defined by Eqs. (4.1), (4.2a), (4.3), and A = 1 if the governing equations (4.1), (4.2b), and (4.3) are used. In the functional (4.4), the function $v = v(x,y,t_0)$ must belong to the class of admissible functions, i.e., it must be continuous and have piecewise continuous first derivatives in D, and furthermore it must satisfy the principal or Dirichlet boundary conditions. This latter requirement, of course, only applies if the governing equations

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† If the case of varying free-stream velocity is included.

(4.1), (4.2a), and (4.3) are considered.

It can be shown [2] that that function $v(x,y,t_0)$ which minimizes the given functional (4.4), with the proper choice of the constant A, is also a solution to the field problem defined by Eqs. (4.1), (4.2a) or (4.2b), and (4.3). The finite element method as applied to the functional (4.4) has been described in the literature [2,8,9], and will not be considered any further in this report.

4.1 <u>The Velocity Potential</u>

Comparison of Eqs. (2.8b), (3.8), (3.13), and (3.4), with Eqs. (4.1), (4.2a), (4.2b), and (4.3) indicate that the solution to the velocity potential problem considered in Sections (2.1) and (3.1) can be obtained using the above analysis and making the following choice for the variables g, h, and z, respectively,

$$g(x,y,t_0) = -Ux - Vy \quad \text{on } S_0 \quad , \tag{4.5a}$$

$$h(x,y,t_0) = -\bar{q}_{uniform} \cdot \bar{n}_0 \quad \text{on } S_0 \quad , \tag{4.5b}$$

$$z(x,y,t_0) = -\bar{u}_1(t_0) \cdot \bar{n}_1 \quad \text{on } S_1 \quad . \tag{4.5c}$$

Substituting the expressions (4.5b), (4.5c) into the functional (4.4), and applying the finite element technique to this functional yields the solution for $\phi = \phi(x,y,t_0)$.

4.2 The Stream Function

Comparison of Eqs. (2.12b), (3.16), (3.17b), and (3.26), with Eqs. (4.1), (4.2a), (4.2b), and (4.3) indicate that the solution to the stream function problem considered in Sections (2.2) and (3.2) can be obtained using the above analysis and making the following choice for the variables g, h, and z, respectively,

$$g(x,y,t_0) = Vx - Uy \quad \text{on } S_0 \quad , \tag{4.6a}$$

$$h(x,y,t_0) = -(\bar{q}_{uniform} \times \bar{n}_0) \cdot \bar{k} \quad \text{on } S_0 \quad , \tag{4.6b}$$

$$z(x,y,t_0) = a_1 - \int_0^S \bar{u}_1 \cdot \bar{n}_1 \, ds \quad \text{on } S_1 \quad . \tag{4.6c}$$

Substituting the expressions (4.6b) and (4.6c) into the functional (4.4), and applying the finite element technique to this functional yields the solution for $\psi = \psi(x,y,t_0)$.

5. ENTRAINED MASS OR INERTIA

In moving body problems such as described in the previous sections, the pressure can be calculated from Eq. (2.9),

$$\frac{p}{\rho} - \frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 = f(t) \quad , \tag{2.9}$$

which can be rearranged as

$$p = -\frac{\rho}{2}(\nabla\phi)^2 + \rho\frac{\partial\phi}{\partial t} + \rho f(t) \quad , \tag{5.1}$$

or as

$$p = p_v + p_a + p_0 \tag{5.2}$$

where the first right-hand-side term is interpreted as the pressure related to velocity, and the second term as the pressure related to acceleration since it disappears in the absence of acceleration.

The integral of p_v around the body surface gives the force corresponding to steady-state conditions, whereas the integral of p_a gives the 'entrained mass' force. For a non-lifting body in an irrotational stream, the latter acceleration force divided by the instantaneous acceleration gives a constant dependent on the shape and orientation of the body known as the entrained mass.

The finite element approaches outlined earlier in this report can be used to solve entrained mass problems since Eq. (5.1) can be evaluated at a given point on the body surface at a particular time t_1 with $\nabla\phi$ and $\frac{\partial\phi}{\partial t}$ being obtained as follows:

(1) $\nabla\phi$ can be obtained from the finite element solution at the given time t_1 by either

a) using ϕ as the unknown and deriving $\nabla\phi$ from the resultant solution, or

b) using a higher-order shape function solution with ϕ and the elements of $\nabla\phi$ as unknowns.

(2) $\frac{\partial\phi}{\partial t}$ at t_1 can be derived by using the ϕ solutions for $t = t_1 - \Delta t$ and $t = t_1$ and extrapolating from $\frac{\partial\phi}{\partial t}$ at $(t_1-\Delta t)$ using a suitable formula.

Integration of the pressure around the surface allows the acceleration force and hence the entrained mass or inertia to be derived.

The stream function solutions given previously can be used

for two-dimensional entrained mass problems whereas the velocity potential methods can be used for both two- and three-dimensional problems.

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